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NONLINEAR FEEDBACK SOLUTION FOR MINIMUM-TIME INJECTION INTO CIRCULAR ORBIT WITH CONSTANT THRUST ACCELERATION MAGNITUDE

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ABSTRACT

The instantaneous thrust-direction for a rocket vehicle to perform a minimum-time injection into a circular orbit of prescribed radius is determined as a function of instantaneous distance, and radial and tangential velocity relative to the attracting center. A nonlinear feedback control law for the instantaneous thrust-direction is derived which is based on the approximation that the gravity vector and the vehicle thrust acceleration magnitude during the maneuver are to be constant at values intermediate between their present and expected terminal values. The control law is shown to depend only on two dimensionless functions of the three relevant state variables, so that the solution is, in effect, expressed in a reduced state space of two dimensions. The optimal thrust-direction is defined analytically and graphically as a function on the reduced state space.

The open-loop solution to the minimum-time transfer problem is the well-known linear tangent law. The new contributions here are (1) showing

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that the solution depends on only two dimensionless functions of state and (2) putting the solution in the form of a feedback law which depends on these two functions.

For a maneuver spanning a considerable arc around the attracting center (up to about 40°), the solution may be used directly as a suboptimal control or to give starting values for an iterative solution of the true inverse-square gravity problem. More appropriately, it may be used to determine terminal thrusts to circularize a near-circular orbit near the desired altitude or for intermittent thrusts to maintain a satellite in a desired circular orbit.

INTRODUCTION

The injection of a rocket vehicle into a circular orbit of radius r in a gravity field $\frac{\mu}{r^2}$ involves simultaneously nulling the radial velocity and achieving tangential velocity $U = \sqrt{\frac{\mu}{r}}$ at radius r. A feedback control to achieve these conditions may be derived by assuming that at each instant (1) the gravity vector is to be constant in the region of the remaining maneuver at some value intermediate between its value at the present position and its value at the expected point of injection, and (2) the magnitude of thrust acceleration is to be constant at a value intermediate between its present value and its value at the expected time of injection. The assumption of constant gravity is reasonably good for nearly circular orbits, if the maneuver time is short enough that the angular distance traveled around the attracting center is less than 30° to 40° [Ref. 1]. With the added assumption that the vehicle position and velocity lie in the plane of the desired orbit, the terminal phase of the injection maneuver may be approximated by the planar problem of choosing thrust direction for minimum-time transfer to specified altitude and horizontal velocity, using a constant magnitude thrust acceleration, in a gravity field constant in direction and magnitude.

Let y_o, u_o, v_o, t_o, β_o and h, U, 0, t_f, β_f be initial and final values of altitude, horizontal and vertical velocity, time, and thrust direction angle above the horizontal. Let a and g be the magnitudes of vehicle thrust acceleration and gravity. The present optimal thrust direction β_o will be related to present state y_o, u_o, v_o through the equations:

$$\eta = f_1(\beta_o, \beta_f; \tfrac{g}{a}) \qquad \text{where } \eta = \frac{2a(h-y_o)}{(U-u_o)^2 + v_o^2}$$

$$\tan\gamma = f_2(\beta_o, \beta_f; \tfrac{g}{a}) \qquad \tan\gamma = \frac{v_o}{U-u_o}$$

The feedback law will involve computing η and $\tan\gamma$ from current state and desired h and U, inverting the functions f_1 and f_2 to obtain β_o, β_f, and using β_o for control. Now β_o will depend only on the two dimensionless quantities η, γ, instead of on the three physical quantities y_o, u_o, v_o. Similarly, β_f and the normalized time-to-injection

$$\tau = \frac{a(t_f - t_o)}{v_o} = f_3(\beta_o, \beta_f; \tfrac{g}{a})$$

will depend only on η and γ. Thus, in the y_o, u_o, v_o state space, a parabola specified by the two parameters η, γ will be a locus of constant β_o, constant β_f, and constant τ. The solution to a problem stated in the three-space y_o, u_o, v_o will be expressed in the two-space η, γ.

MINIMUM-TIME TRANSFER

IN THE REDUCED STATE SPACE η, γ

Take an earth-centered inertial frame with vertical along the present value of the intermediate gravity vector $\vec{g}$. Let x, y, u, v denote horizontal and vertical components of position and velocity at time t; let β denote thrust elevation above the horizontal; and let subscripts o and f denote initial and final values of these quantities, as shown in Fig. 1. The motion satisfies

$$\dot{x} = u \tag{1}$$

$$\dot{y} = v \tag{2}$$

$$\dot{u} = a \cos\beta \tag{3}$$

$$\dot{v} = a \sin\beta - g \tag{4}$$

where a and g are magnitudes of the intermediate thrust acceleration and gravity defined in Appendix A.

The optimal thrust direction β is chosen to minimize the Hamiltonian

$$\mathcal{H} = 1 + \lambda_x u + \lambda_y v + \lambda_u a \cos\beta + \lambda_v (a \sin\beta - g) \tag{5}$$

for values of the costate vector satisfying the Euler-Lagrange equations

$$\dot{\lambda}_u = -\lambda_x \tag{6}$$

$$\dot{\lambda}_v = -\lambda_y \tag{7}$$

$$\dot{\lambda}_x = 0 \tag{8}$$

$$\dot{\lambda}_y = 0 \tag{9}$$

and values of state satisfying (1) - (4). Integrating (6) - (9),

$$\lambda_x = \text{constant} \tag{10}$$

$$\lambda_y = \text{constant} \tag{11}$$

$$\lambda_u = \lambda_{u_o} - \lambda_x (t - t_o) \tag{12}$$

$$\lambda_v = \lambda_{v_o} - \lambda_y (t - t_o) \tag{13}$$

For injection into a horizontal trajectory at altitude h with velocity U , we prescribe terminal conditions

$$y_f = h \tag{14}$$

$$u_f = U \tag{15}$$

$$v_f = 0 \tag{16}$$

Since the downrange distance x_f at injection is free, $\lambda_x = 0$, so λ_u = constant.

The thrust direction which minimizes $\mathcal{H}$ is given by

$$\tan \beta = \frac{\lambda_v}{\lambda_u} = \frac{\lambda_{v_o} - \lambda_y (t - t_o)}{\lambda_u} \tag{17}$$

or

$$\tan \beta_f = \tan \beta_o - c(t_f - t_o) \tag{18}$$

where

$$\tan \beta_o = \frac{\lambda_{v_o}}{\lambda_u} \tag{19}$$

$$c = \frac{\lambda_y}{\lambda_u} \tag{20}$$

The state history for minimum-time injection into a horizontal trajectory starting from rest and in the absence of gravity is given in Chapter 2 of Reference 2 . The solution is there obtained by integrating (1) - (4) with β as independent variable, using the relation

$$\frac{d\beta}{dt} = - c \cos^2 \beta \tag{21}$$

from (17) and (20) to change from t to β as independent variable. Adding terms due to initial velocity and gravity yields

$$u_f = \frac{a}{c} \ln \frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f} + u_o \tag{22}$$

$$v_f = \frac{a}{c} (\sec\beta_o - \sec\beta_f) - g(t_f - t_o) + v_o \tag{23}$$

$$y_f = \frac{a}{2c^2} \left[(\tan\beta_o - \tan\beta_f)\sec\beta_o - (\sec\beta_o - \sec\beta_f)\tan\beta_f - \ln\frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f} \right] - \frac{1}{2} g (t_f - t_o)^2 + v_o (t_f - t_o) + y_o \tag{24}$$

Imposing terminal conditions (14) - (16) on (22) - (24) yields three equations which, together with (18), constitute a set of four equations in unknowns β_o, β_f, $t_f - t_o$ and c .

Following a procedure suggested in Reference 2, we obtain a pair of equations defining β_o and β_f implicitly as functions of the dimensionless variables

$$\eta = \frac{2a(h - y_o)}{v_o^2 + (U - u_o)^2} \tag{25}$$

$$\tan\gamma = \frac{v_o}{U - u_o} \tag{26}$$

The functions $\eta = f_1(\beta_o, \beta_f; \frac{g}{a})$ and $\tan\gamma = f_2(\beta_o, \beta_f; \frac{g}{a})$ are stated below and are derived in Appendix B.

$$\eta = \left[\frac{v_o^{\;2}}{2a(h - y_o)} + \frac{(U - u_o)^2}{2a(h - y_o)} \right]^{-1} \tag{27}$$

where

$$\frac{2a(h - y_o)}{v_o^{\;2}} = \frac{1}{\left[\frac{g}{a}(\tan\beta_o - \tan\beta_f) - (\sec\beta_o - \sec\beta_f)\right]^2} \left\{[A] - \frac{g}{a}(\tan\beta_o - \tan\beta_f)^2\right\} + \frac{2(\tan\beta_o - \tan\beta_f)}{\frac{g}{a}(\tan\beta_o - \tan\beta_f) - (\sec\beta_o - \sec\beta_f)} \tag{28}$$

$$\frac{2a(h - y_o)}{(U - u_o)^2} = \frac{1}{\left[\ln\frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f}\right]^2} \left\{[A] + \frac{g}{a}(\tan\beta_o - \tan\beta_f)^2 - 2(\tan\beta_o - \tan\beta_f)(\sec\beta_o - \sec\beta_f)\right\} \tag{29}$$

and

$$\tan\gamma = \frac{\frac{g}{a}(\tan\beta_o - \tan\beta_f) - (\sec\beta_o - \sec\beta_f)}{\ln\frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f}} \tag{30}$$

The symbol [A] represents the term in square brackets in (24). The normalized time-to-injection is

$$\tau = \frac{a(t_f - t_o)}{v_o} = \frac{\tan\beta_o - \tan\beta_f}{\frac{g}{a}(\tan\beta_o - \tan\beta_f) - (\sec\beta_o - \sec\beta_f)} \tag{31}$$

Note that η and γ determine β_o and β_f through (27) and (30), and β_o and β_f determine τ through (31). Thus, the optimum thrust direction angle above the horizontal, β_o , is given as a function of η and γ . This is the feedback law for minimum-time injection into a horizontal trajectory for a vehicle with constant thrust acceleration magnitude in a gravity field constant in direction and magnitude.

For the case $\frac{a}{g} = 3$, Fig. 2 shows minimum-time paths and contours of constant thrust-direction angle on an η versus γ plot. Figure 3 shows the same minimum-time paths as Fig. 2, with the paths intersecting contours (isochrones) of constant dimensionless time-to-injection, again for $\frac{a}{g} = 3$.

To illustrate the use of these charts, suppose that at the radius of the desired orbit the initial altitude-to-be-gained, $h - y_o$, horizontal velocity-to-be-gained, $U - u_o$, and vertical velocity-to-be-lost, v_o, map to point "t" ($\gamma = 37^o$, $\frac{1}{\eta} = .35$) on Fig. 2. The optimal thrust direction at "t" is $\beta_o = 80^o$, and the extremal path is that labeled $\beta_f = -80^o$. Following this path, β_o is reduced to 40^o at "y", to 0^o at "x", and finally to $\beta_o = \beta_f = -80^o$ at the injection point "h". The values of dimensionless time-to-injection along this path may be read from Fig. 3.

The heavy line labeled "locus of fixed-point extremals" on Fig. 2 and "terminal manifold" on Fig. 3 is the locus on which all extremals terminate. It is the locus for which extremal trajectories are fixed points in (η, γ) space, and the thrust direction is constant at a value such that the vector sum of thrust and gravity is along the velocity-to-be-gained vector. Details of the mapping from (β_o, β_f) to (η, γ) are discussed in Appendix C.

CONCLUSION

A nonlinear feedback law has been obtained for controlling thrust direction to produce minimum-time injection of a spacecraft into circular orbit. This law depends only on two dimensionless quantities which can be determined from three physical quantities: (1) distance from the attracting center, (2) radial velocity, and (3) tangential velocity.

- - - - -

REFERENCES

1. Seifert, H. S. (Ed.), Space Technology, John Wiley and Sons, Inc., New York, 1959, Chapters 5, 10.

2. Bryson, A. E., Jr., and Ho, Y. C., Optimal Programming, Estimation, and Control (to be published by Blaisdell Press).

APPENDIX A

Intermediate Value of Gravity Vector and Thrust Acceleration Magnitude

We choose the magnitude of the intermediate gravity vector g so that the increment in potential energy corresponding to ascent from initial radial distance r_o to final radial distance r_f in the true $\frac{\mu}{r^2}$ gravity field will be the same as is obtained by ascent through height $r_f - r_o$ in a constant gravity field g.

$$(r_f - r_o) g = \frac{\mu}{r_f} - \left(- \frac{\mu}{r_o} \right) \tag{A-1}$$

$$g = \frac{\mu}{r_o r_f} \tag{A-2}$$

This is an arbitrary but reasonable choice of g.

We choose the direction of the intermediate gravity vector and the magnitude of the intermediate thrust acceleration by the following iteration:

1. Let the first estimate of intermediate $\vec{g}$ have magnitude (A-2) and direction downward along the present vertical. With the xy coordinate frame so defined, evaluate η, γ and solve for β_o, β_f by a Newton-Raphson iteration starting from β_o, β_f values stored in an η, γ grid for nominal $\frac{a}{g}$. Compute $t_f - t_o$ from (31) and c from (18). Compute intermediate a, assuming constant thrust T, and mass flow rate $\dot{m}$.

$$a = \frac{T}{2} \left[\frac{1}{m(t_o)} + \frac{1}{m(t_o) - (t_f - t_o)\dot{m}} \right] \tag{A-3}$$

2. Compute

$$x_f - x_o = \frac{a}{c^2} \left(\sec\beta_o - \sec\beta_f - \tan\beta_f \ell n \frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f} \right) + u_o (t_f - t_o) \tag{A-4}$$

and estimate the angle to injection

$$\theta = \frac{x_f - x_o}{\left(\frac{r_o + r_f}{2}\right)} \tag{A-5}$$

The formula for $x_f - x_o$ is derived in Reference 2, Chapter 2.

3. Let the next estimate of the direction of $\vec{g}$ be halfway between the present vertical and the vertical at the estimated point of injection, i.e., bisecting the angle θ .

4. In the new xy frame so defined, evaluate η , γ , and repeat steps one through four.

The need for care in selecting intermediate $\vec{g}$ and a can be shown by a simplified problem. Suppose that gravity were truly constant in direction and magnitude at value $\vec{g}$, and thrust acceleration were constant at a , and that control were based on the correct direction of $\vec{g}$, but incorrect magnitudes $\hat{g}$ and $\hat{a}$. Further suppose that the initial horizontal velocity were the desired value, so that the problem is the purely vertical one of nulling vertical velocity at the desired altitude.

The control consists of switching curves. With perfect knowledge of a and g , only one switch is required. But with imperfect knowledge, the vehicle under true net upward acceleration $a - g$ or downward acceleration $a + g$ cannot follow the switching curves (based on nominal $\hat{a} - \hat{g}$ or $\hat{a} + \hat{g}$) to the desired state. Many switches are required, the number increasing with the deviation of $(\hat{a}, \hat{g})$ from (a, g), and decreasing with the width of a tolerance zone along the switching curve.

If the vehicle must reorient to thrust, this is expensive in attitude control, and interposes periods during which no thrust can be applied while the vehicle is rotated through 180°. Even if no reorientation is required, time is wasted while state, following parabolas based on actual $a - g$ and $a + g$, departs and returns in short arcs from the switching curves based on $\hat{a} - \hat{g}$ and $\hat{a} + \hat{g}$.

APPENDIX B

Initial and Final Thrust Direction

Expressed Implicitly As Functions of η and γ

The equations

$$U - u_o = \frac{a}{c} \ln \frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f} \tag{B-1}$$

$$-v_o = \frac{a}{c}(\sec\beta_o - \sec\beta_f) \tag{B-2}$$

$$\begin{aligned} h - y_o = \frac{a}{2c^2} \Bigg[&(\tan\beta_o - \tan\beta_f)\sec\beta_o - (\sec\beta_o - \sec\beta_f)\tan\beta_f - \\ &\ln\frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f}\Bigg] - \frac{1}{2}g(t_f - t_o)^2 + v_o(t_f - t_o) \end{aligned} \tag{B-3}$$

$$c = \frac{\tan\beta_o - \tan\beta_f}{t_f - t_o} \tag{B-4}$$

are to be solved for β_o, β_f, $t_f - t_o$, and c.

From (B-2) and (B-4), the normalized time-to-injection τ is

$$\frac{a(t_f - t_o)}{v_o} = \frac{1}{\frac{g}{a} - \left(\frac{\sec\beta_o - \sec\beta_f}{\tan\beta_o - \tan\beta_f}\right)} = \frac{\tan\beta_o - \tan\beta_f}{\frac{g}{a}(\tan\beta_o - \tan\beta_f) - (\sec\beta_o - \sec\beta_f)} \tag{B-5}$$

From (B-1) and (B-4)

$$U - u_o = \frac{a(t_f - t_o)}{\tan\beta_o - \tan\beta_f} \ln\frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f} \tag{B-6}$$

Dividing by v_o and using (B-5) and definition (26),

$$\tan\gamma = \frac{v_o}{U - u_o} = \frac{\frac{g}{a}(\tan\beta_o - \tan\beta_f) - (\sec\beta_o - \sec\beta_f)}{\ln\frac{\tan\beta_o + \sec\beta_o}{\tan\beta_f + \sec\beta_f}} \tag{B-7}$$

Substituting (B-4) in (B-3) and multiplying the resulting equation by $\frac{2a}{v_o^2}$,

$$\frac{2a(h-y_o)}{v_o^2} = \frac{a^2(t_f-t_o)^2[A]}{v_o^2(\tan\beta_o-\tan\beta_f)^2} - \frac{ag}{v_o^2}(t_f-t_o)^2 + \frac{2a}{v_o}(t_f-t_o) \tag{B-8}$$

where $[A]$ is the term in brackets in (24). Substituting (B-5) in (B-8),

$$\frac{2a(h-y_o)}{v_o^2} = \frac{1}{\left[\frac{g}{a}(\tan\beta_o-\tan\beta_f)-(\sec\beta_o-\sec\beta_f)\right]^2}\left\{[A]-\frac{g}{a}(\tan\beta_o-\tan\beta_f)^2\right\} + \frac{2(\tan\beta_o-\tan\beta_f)}{\frac{g}{a}(\tan\beta_o-\tan\beta_f)-(\sec\beta_o-\sec\beta_f)} \tag{B-9}$$

Multiplying (B-9) by the square of (B-7),

$$\frac{2a(h-y_o)}{(U-u_o)^2} = \frac{1}{\left[\ell n\,\frac{\tan\beta_o+\sec\beta_o}{\tan\beta_f+\sec\beta_f}\right]^2}\left\{[A]+\frac{g}{a}(\tan\beta_o-\tan\beta_f)^2 - 2(\tan\beta_o-\tan\beta_f)(\sec\beta_o-\sec\beta_f)\right\} \tag{B-10}$$

Then

$$\eta = \left[\frac{v_o^2}{2a(h-y_o)} + \frac{(U-u_o)^2}{2a(h-y_o)}\right]^{-1} \tag{B-11}$$

is evaluated from (B-9) and (B-10). The functions f_1, f_2, f_3 mentioned in the Introduction are given by (B-11), (B-7), and (B-5), respectively.

APPENDIX C

Extremals, Loci of Constant Control, Isochrones, and the Terminal Manifold in the Reduced State Space

As present time t_o approaches final time t_f, the present thrust direction β_o approaches the final thrust direction β_f, following the linear tangent law

$$\tan\beta_f = \tan\beta_o - c(t_f - t_o) \tag{C-1}$$

Thus, in β_o, β_f space (see Fig. 4) the extremal (minimum-time) trajectories are paths β_f = constant, terminating on the line $\beta_o = \beta_f$, and the loci of constant control are lines β_o = constant. This grid of extremals and constant control loci is mapped from β_o, β_f (Fig. 4) to η, γ (Fig. 2) by the relations $\eta = f_1(\beta_o, \beta_f; \frac{g}{a})$ and $\tan\gamma = f_2(\beta_o, \beta_f; \frac{g}{a})$. A comparison of Figs. 2 and 4 shows that the mapping is topographic, but not conformal. Corresponding points on the two figures have been labeled with corresponding letters. In cases where the relation of β_o = constant and β_f = constant curves in Fig. 2 are obscured by crowding, it is convenient to refer to Fig. 4 to see what portions of β_o, β_f space are mapped into small regions of η, γ space.

The case $c = 0$ corresponds to $\lambda_y = 0$ (see (20)), i.e., to free terminal altitude. The extremals for this case are simply fixed points on the line $\beta_o = \beta_f$. Since all extremals terminate on the line $\beta_o = \beta_f$, we refer to it, and its image in η, γ space, as the "terminal manifold."

With terminal altitude free, the constant thrust-direction angle β and time-to-injection $t_f - t_o$ satisfy

velocity-to-be-gained = (net acceleration) $\bullet$ $(t_f - t_o)$:

$$(U - u_o)\hat{\imath} - v_o\hat{\jmath} = [a\cos\beta\,\hat{\imath} + (a\sin\beta - g)\hat{\jmath}](t_f - t_o) \tag{C-2}$$

so that

$$\tan\gamma = \frac{v_o}{U - u_o} = \frac{1 - \frac{a}{g}\sin\beta}{\frac{a}{g}\cos\beta} \qquad \text{(C-3)}$$

$$v_o^2 + (U - u_o)^2 = (t_f - t_o)^2 (g^2 + a^2 - 2ag\sin\beta) \qquad \text{(C-4)}$$

To reach the desired altitude under constant thrust,

$$h = y_o + v_o(t_f - t_o) + \frac{1}{2}(t_f - t_o)^2 (a\sin\beta - g) \qquad \text{(C-5)}$$

Substituting v_o from (C-2),

$$h - y_o = \frac{1}{2}(t_f - t_o)^2 (g - a\sin\beta) \qquad \text{(C-6)}$$

From (C-4) and (C-6),

$$\eta = \frac{2a(h - y_o)}{v_o^2 + (U - u_o)^2} = \frac{\frac{a}{g}(1 - \frac{a}{g}\sin\beta)}{1 + \left(\frac{a}{g}\right)^2 - 2\frac{a}{g}\sin\beta_o} \qquad \text{(C-7)}$$

From (C-2)

$$\tau = \frac{a(t_f - t_o)}{v_o} = \frac{1}{\frac{g}{a} - \sin\beta} \qquad \text{(C-8)}$$

The terminal manifold represents those trajectories in which only terminal velocity was constrained, and the desired altitude was reached, by coincidence, simultaneously with the desired velocity. Equations (C-3), (C-7), and (C-8) rather than (B-7), (B-11), and (B-5) must be used on the terminal manifold, since the latter expressions are indeterminate for $\beta_o = \beta_f$.

To compute contours of constant τ, it is convenient to introduce new variables

$$\sigma = \beta_o + \beta_f \qquad \text{(C-9)}$$

$$\delta = \beta_o - \beta_f \tag{C-10}$$

in terms of which (B-5) becomes

$$\tau = \frac{1}{\frac{g}{a} - \frac{\sin\frac{\sigma}{2}}{\cos\frac{\delta}{2}}} \tag{C-11}$$

We define

$$k\left(\frac{\delta}{2}, \frac{\sigma}{2}\right) \triangleq \frac{\sin\frac{\sigma}{2}}{\cos\frac{\delta}{2}} = \frac{g}{a} - \frac{1}{\tau} \tag{C-12}$$

so that, for given $\frac{g}{a}$, contours of constant τ are contours of constant k in β_o , β_f space, which can then be mapped to η , γ space by

$$\eta = f_1(\beta_o , \beta_f ; \frac{g}{a})$$

and

$$\tan\gamma = f_2(\beta_o , \beta_f ; \frac{g}{a}) \quad .$$

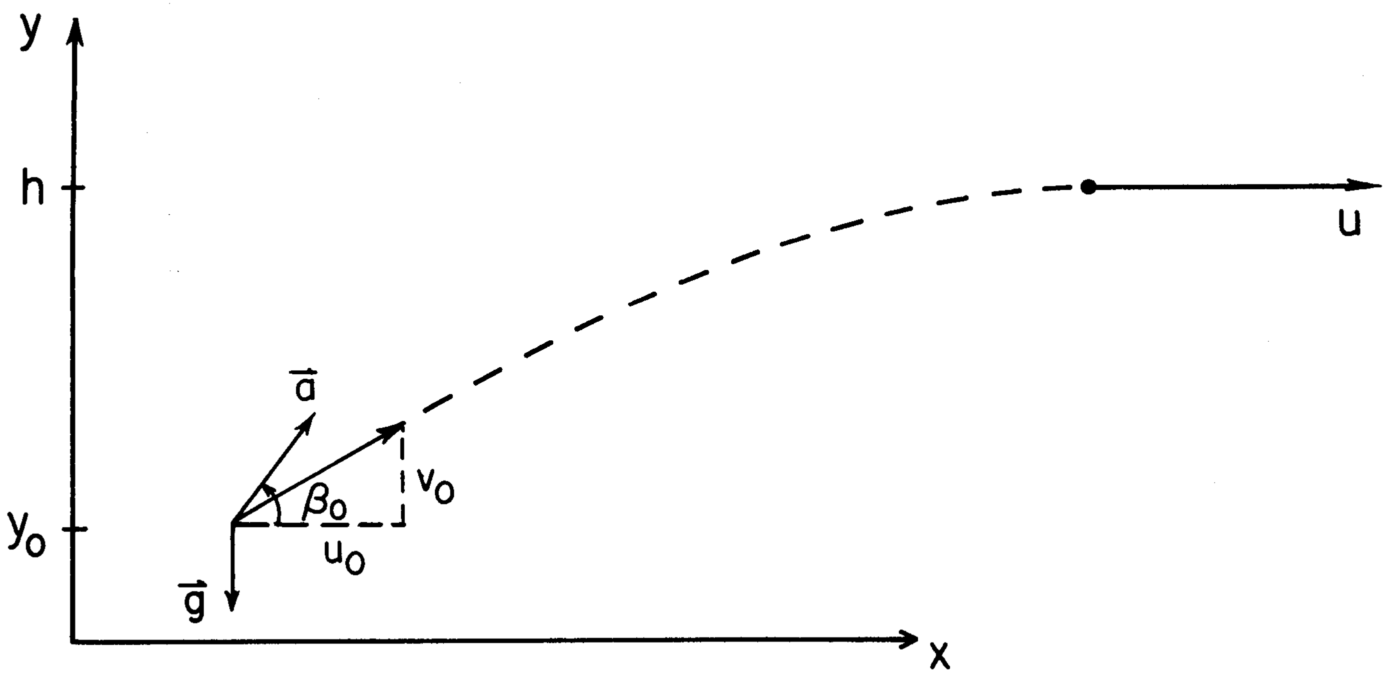

FIGURE 1 INJECTION INTO HORIZONTAL TRAJECTORY

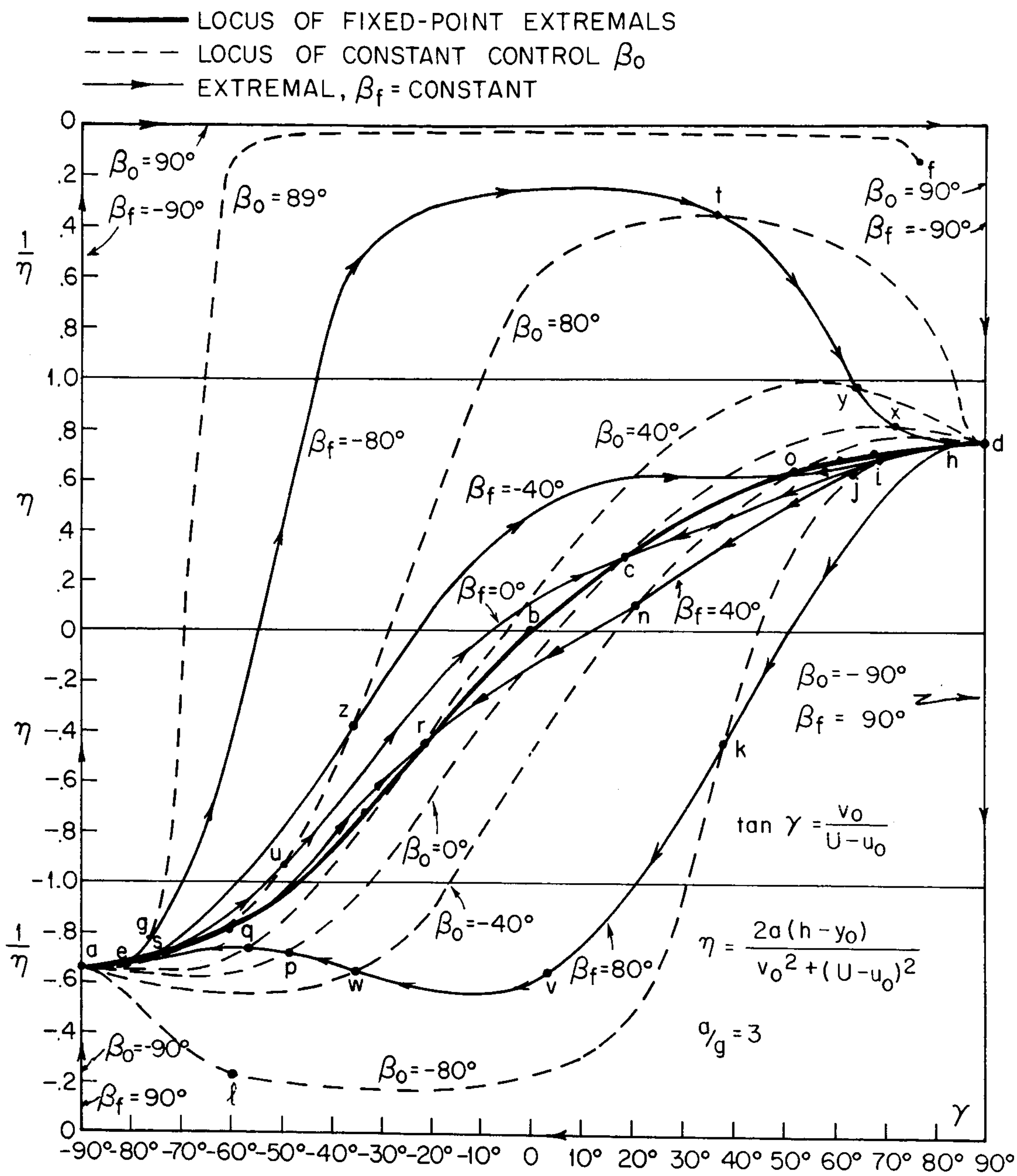

FIGURE 2 EXTREMALS AND CONTOURS OF CONSTANT CONTROL IN η, γ SPACE.

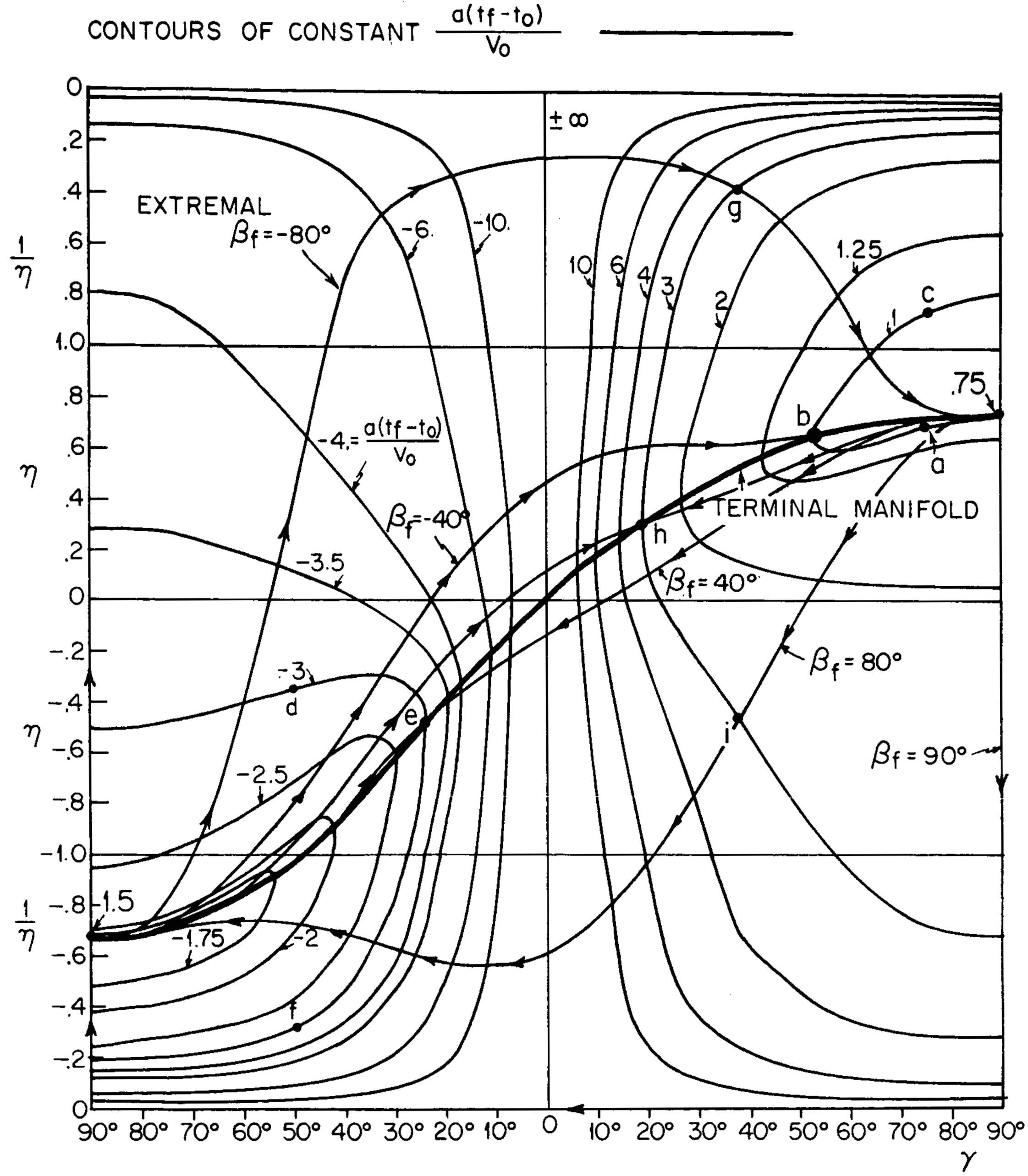

FIGURE 3 ISOCHRONES AND EXTREMALS
IN η, γ SPACE

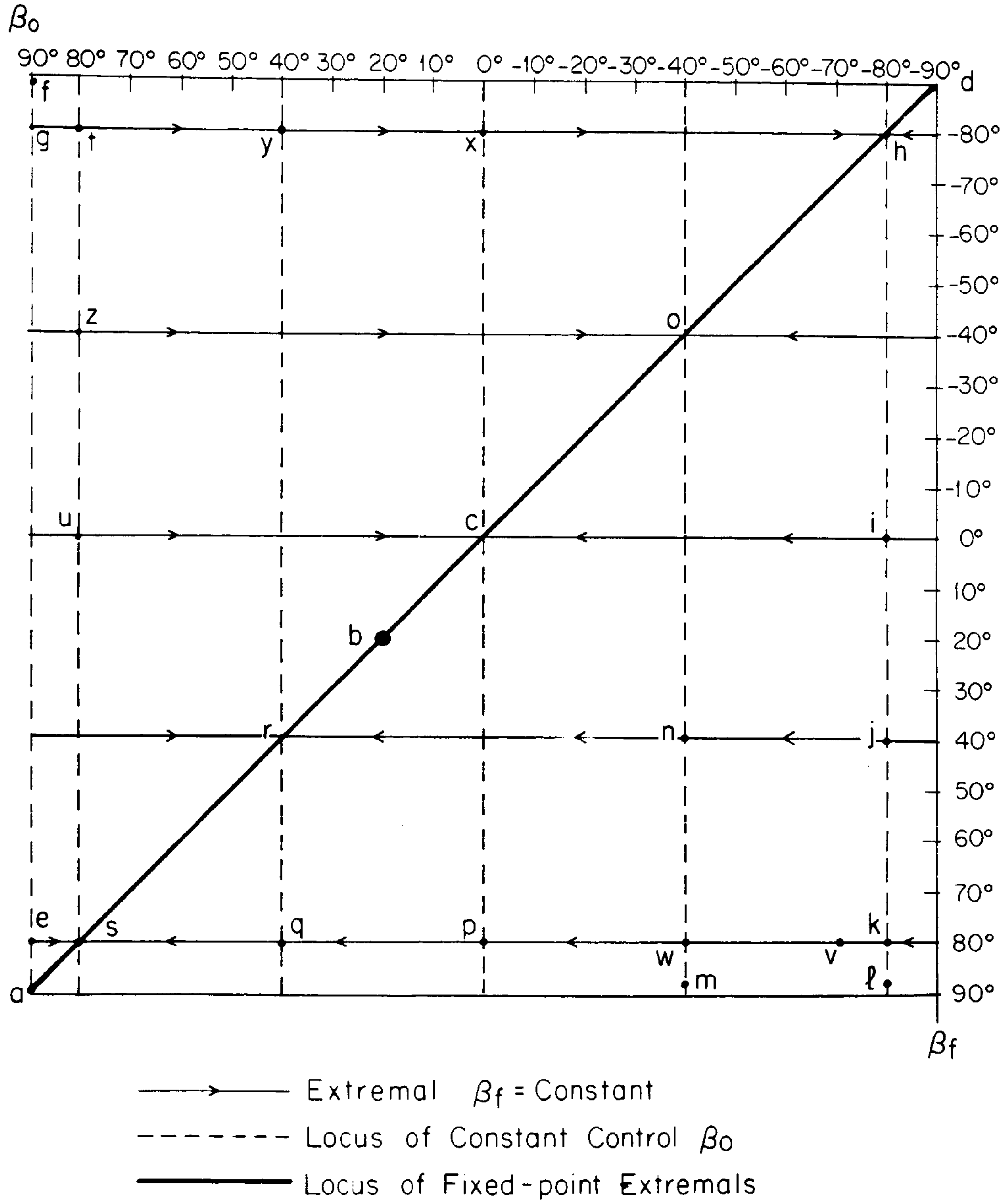

FIGURE 4 EXTREMALS AND CONSTANT CONTROL LOCI IN β_o, β_f SPACE.

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13. ABSTRACT The instantaneous thrust-direction for a rocket vehicle to perform a minimum-time injection into a circular orbit of prescribed radius is determined as a function of instantaneous distance, and radial and tangential velocity relative to the attracting center. A nonlinear feedback control law for the instantaneous thrust-direction is derived which is based on the approximation that the gravity vector and the vehicle thrust acceleration magnitude during the maneuver are to be constant at values intermediate between their present and expected terminal values. The control law is shown to depend only on two dimensionless functions of the three relevant state variables, so that the solution is, in effect, expressed in a reduced state space of two dimensions. The optimal thrust-direction is defined analytically and graphically as a function on the reduced state space.

The open-loop solution to the minimum-time transfer problem is the well-known linear tangent law. The new contributions here are (1) showing that the solution depends on only two dimensionless functions of state and (2) putting the solution in the form of a feedback law which depends on these two functions.

For a maneuver spanning a considerable arc around the attracting center (up to about 40°), the solution may be used directly as a suboptimal control or to give starting values for an iterative solution of true inverse-square gravity problem. More appropriately, it may be used to determine terminal thrusts to circularize a near-circular orbit near the desired altitude or for intermittent thrusts to maintain a satellite in a desired circular orbit.

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